Math 259A Lecture 5 Notes

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1 The GNS Construction

1.1 The idea: turning abstract C^*-algebras in to concrete ones

Let M be a C^*-algebra (with 1_M) and $x = x^* \in M_h$. Then the C^*-algebra generated by x can be identified with $\mathrm{Spec}(x)$. Then denote

$$x_+ = f_+(x), \qquad x_- = f_-(x),$$

where

$$f_+(x) = \max\{x, 0\}, \qquad f_-(x) = -\min\{x, 0\}.$$

Then $x = x_+ - x_-$, $x_+ x_- = 0$, and we can also define $|x| = x_+ + x_- = (x^2)^{1/2}$. If $\mathrm{Spec}(x) \subseteq [0, \infty)$, then we can define $\sqrt{x}$ using functional calculus.

Lemma 1.1. *If $x = x^* \in M$ and $\|1 - x\| \leq 1$, then $\mathrm{Spec}(x) \subseteq [0, \infty)$. Conversely, if $\mathrm{Spec}(x) \subseteq [0, \infty)$, then $\|1 - x\| \leq 1$.*

Proof. This follows from functional calculus. □

Lemma 1.2. *Let S, T be elements of a Banach algebra. Then $\mathrm{Spec}(ST) \cup \{0\} = \mathrm{Spec}(TS) \cup \{0\}$.*

Proof. If $\lambda \neq 0$ and $TS - \lambda 1$ has inverse u, then $TSu = \lambda u + 1$, so

$$(ST - \lambda 1)(SuT - 1) = STSuT - ST - \lambda - \lambda SuT + \lambda 1 = \lambda 1.$$ □

Recall: We want to show that if M is a C^*-algebra, there is an isometric embedding $\pi : M \to B(H)$, where H is a Hilbert space; that is, every abstract C^*-algebra is a concrete C^*-algebra. To get the isometry property, we only need $\|\pi(x)\|^2 = \|x\|^2$, which means we need $\|\pi(x^*x)\| = \|x^*x\|$. This is the spectral radius of x^*x and $\pi(x^*x)$, so we need only show that π is injective.

Suppose we have that if $x \neq 0$ then there is a $\pi_x : M \to B(H_x)$ with $\pi_x(x) \neq 0$. Then we can take $\bigoplus_x \pi_x : M \to \bigoplus B(H_x)$. So we only need to find π_x for each x. To find π_x, we claim that all we need is a functional φ which has $\varphi(y^*y) \geq 0$ for $y \in M$ and $\varphi(x^*x) \neq 0$. Then we will be able to get a Hilbert space by looking at M itself with the inner product $\langle y, x \rangle_\varphi = \varphi(y^*x)$ (this is a Hilbert space if we mod out by some equivalence relation). To find a functional φ, we will need to use Hahn-Banach.

1.2 Characterizing positive elements in a C^*-algebra

Proposition 1.1 (Positive elements in C^*-algebras)**.** *Let M be a C^*-algebra, and let $x = x^* \in M_h$. The following are equivalent:*

1. $\mathrm{Spec}(x) \subseteq [0, \infty)$.
2. *$x = y^*y$ for some $y \in M$.*
3. *$x = h^2$ for some $h \in M_h$.*

Also, if we denote M_+ to be the set of elements satisfying these conditions, then M_+ is a closed, convex cone in M_h ($x \in M_+, \lambda \geq 0 \implies \lambda x \in M_+$ and $x, y \in M_+ \implies x + y \in M_+$). Moreover, $M_+ \cap (-M_+) = \{0\}$.

Proof. Let P be the set of elements in M_h satisfying condition 1.

(1) $\implies$ (3): Take $h = \sqrt{x}$ by functional calculus.

(3) $\implies$ (2): Take $y = y^* = h$.

(3) $\implies$ (1): Since h is self-adjoint, $\mathrm{Spec}(h) \subseteq \mathbb{R}$. Then we have $\mathrm{Spec}(h^2) = (\mathrm{Spec}(h))^2 \subseteq [0, \infty)$.

(2) $\implies$ (3): Write $y^*y = (y^*y)_+ - (y^*y)_- := u^2 - v^2$. Then

$$(yv)^*(yv) = v(y^*y)v = v(u^2 - v^2)v = -v^4$$

has spectrum $\subseteq (-\infty, 0]$. Let $yv = s + it$ with $s, t \in M_h$. Then

$$(yv)(yv)^* = \underbrace{(s - it)(s + it)}_{s^2+t^2} + \underbrace{(s + it)(s - it)}_{s^2+t^2},$$

so if P is a convex cone, then this is in P. Then also $(yv)^*(yv) \in P$ (because $\mathrm{Spec}(TS) \cup \{0\} = \mathrm{Spec}(ST) \cup \{0\}$). So we get that $\mathrm{Spec}((yv)^*(yv)) = 0$, which means that $yv = 0$. So $v = 0$.

To show that P is a cone, we use that for $x \in M_h$, $x \in P \iff \| \|x\|(1 - x)\| \leq 1$ (from the lemma before). This implies that P is closed. On the other hand, if $x \in P$ and $\lambda > 0$, then $\lambda x \in P$ by functional calculus. And if $x, y \in P$ (and now we can assume $\|x\|, \|y\| \leq 1$), then

$$\left\| 1 - \frac{x+y}{2} \right\| \leq \frac{1}{2} \underbrace{\|1 - x\|}_{\leq 1} + \frac{1}{2} \underbrace{\|1 - y\|}_{\leq 1} \leq 1,$$

so $\frac{x+y}{2} \in P$. It follows that P is a closed, convex cone. This completes the proof. □

So from now on, if M is a C^*-algebra, then we denote M_+ to be the cone of positive elements.

1.3 Positive linear functionals

Definition 1.1. A functional $\varphi : M \to \mathbb{C}$ on an involutive algebra is **positive** if $\varphi(x^*x) \geq 0$ for all $x \in M$ and $\varphi(M_+) \subseteq [0, \infty)$.

Definition 1.2. A **state**[1] on an involutive Banach algebra is a positive continuous functional with $\|\varphi\| = 1$.

Proposition 1.2. *If M is an involutive algebra and φ is a positive functional, then M has a pre-Hilbert space structure H_φ with the pre-inner product $\langle x, y \rangle_\varphi = \varphi(y^*x)$.*

Corollary 1.1. *Let $I_\varphi := \{x \in M : \varphi(x^*x) = 0\}$. M/I_φ is an inner product space with $\langle \cdot, \cdot \rangle_\varphi$. The completion is a Hilbert space.*

Proof. We just need I_φ to be a vector space. We have the Cauchy-Schwarz inequality: we have for all $\lambda \in \mathbb{C}$, $\langle x + \lambda y, x + \lambda y \rangle \geq 0$, so the discriminant is ≤ 0; this translates into the desired inequality. Now I_φ is a vector space because we have $\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \left\langle x + i^k y, x + i^k y \right\rangle_\varphi$. $\square$

Proposition 1.3. *I_φ is a left M-ideal. That is, if $y \in I_\varphi$, then $xy \in I_\varphi$ for any $x \in M$.*

Lemma 1.3. *If M is a Banach algebra and $x \in (M)_1$, with $x = 1 + x'$, then the series*

$$h = 1 + \frac{1}{1!} \cdot \frac{1}{2} x' + \frac{1}{2!} \cdot \frac{1}{2} \left(\frac{1}{2} - 1 \right) (x')^2 + \cdots + \frac{1}{n!} \left(\frac{1}{2} - 1 \right) \cdots \left(\frac{1}{2} - (n-1) \right) (x')^n + \cdots$$

is absolutely convergent with $h^2 = x = 1 + x'$. Moreover, if x is self-adjoint, then so is h.

Proposition 1.4. *If M is an involutive Banach algebra and φ is positive on M, then φ is continuous and $\|\varphi\| = \varphi(1)$.*

Proof. By Cauchy-Schwarz, $|\varphi(1x)|^2 \leq \varphi(1)\varphi(x^*x)$. If $y = y^*$ and $\|y\| \leq 1$, then, by the lemma, we have $1 - y = h^2$ with $h = h^*$. Given this representation, $\varphi(x^*x) \leq \varphi(1)$, so $|\varphi(x)| \leq \varphi(1)$. $\square$

Corollary 1.2. *If M is an involutive Banach algebra with 1_M, then the space of states $S(M)$ is convex and weakly compact in $(M^*)_1$.*

Proposition 1.5. *Let M be an involutive Banach algebra, and let φ be positive. Then for all $x, y \in M$,*

$$|\varphi(y^*xy)| \leq \|x\| \varphi(y^*y).$$

Proof. The functional $\varphi_y(x) := \varphi(y^*xy)$ is positive. So $|\varphi_x(x)| \leq \varphi_y(1)\|x\|$. We then have $|\varphi(y^*xy)| \leq \varphi(y^*y)\|x\|$. $\square$

[1] This terminology comes from physics.

1.4 The GNS construction

Corollary 1.3 (GNS[2] construction)**.** *Let M be an involutive Banach algebra, and let φ be positive. Then $\pi_\varphi : M \to \mathcal{B}(M/I)$ given by $\pi_\varphi(x)(\hat{y}) = \widehat{xy}$ is an isometric *-isomorphism of algebras.*

Proof. We have

$$\|\pi_\varphi(x)(\hat{y})\|^2 = \varphi(y^*x^*xy) \leq \|x^*x\|\varphi(y^*y) \leq \|x\|^2\|\hat{y}\|_{M/I_\varphi}.$$

So $\|\pi_\varphi(x)\| \leq \|x\|$, so $\pi_\varphi(x)$ is continuous and extends to all of M/I_φ. We also have

$$\pi_\varphi(x_1x_2) = \pi_\varphi(x_1)\pi_\varphi(x_2),$$

$$\pi_\varphi(x^*) = \pi_\varphi(x)^*.$$

□

Such a map π_φ is called a **representation**.

Proposition 1.6. *If M is a C^*-algebra, then φ is positive if and only if $\|\varphi\| = \varphi(1)$.*

Proof. ($\Longrightarrow$): We have already shown this.

($\Longleftarrow$): If $\varphi(1) = \|\varphi\| = 1$ and $x \geq 0$ in M, suppose $\varphi(x) \ngeq 0$. Then there exists a disc $D \subseteq \mathbb{C}$ centered at some $z_0 \in \mathbb{C}$ such that $\operatorname{Spec}(x) \subseteq D$ but $\varphi(x) \notin D$. Thus, $\operatorname{Spec}(x - z_0 1) \subseteq B_R(0)$, and $x - z_0 1$ is normal. So $\|x - z_0 1\| \leq R$, and

$$|\varphi(x) - z_0| = |\varphi(x - z_0)| \leq \|\varphi\|\|x - z_0\| \leq \|x - z_0\| \leq R.$$

This is a contradiction. □

It remains to show that we can find enough positive linear functionals. We will finish this next time.

[2]This is Gelfand, Naimark, and Segal. Gelfand and Naimark only proved it for the commutative case.